Static solutions in the R^4 brane world

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A simple five-dimensional brane world model is proposed, motivated by M-theory compactified on a six-dimensional manifold of small radius and an S^1/Z_2 of large radius. We include a leading-order higher curvature correction to the tree-level bulk action since in brane world scenarios the curvature scale in the bulk may be comparable to the five-dimensional Planck scale and, thus, higher curvature corrections may become important. As a tractable model of the bulk theory we consider pure gravity including a (Ricci-scalar)^4-correction to the Einstein-Hilbert action. In this model theory, after a conformal transformation to the Einstein frame, we numerically obtain static solutions, each of which consists of a positive tension brane and a negative tension brane. For these solutions, we obtain two relations between the warp factor and the brane tensions. The existence of these relations implies that, contrary to the original Randall-Sundrum model, the so called radion is no longer a zero mode. We conclude that the tension of our brane should be negative and that fine-tuning of the tension of both branes is necessary for a large warp factor to explain the large hierarchy between the Planck scale and the electroweak scale.

I. INTRODUCTION

The idea that our world may be a brane embedded in a higher dimensional spacetime has been attracting a great deal of physical interest. This idea is often called the brane world scenario and, as suggested by Randall and Sundrum [1,2], may be able to explain the large hierarchy between the Planck scale and the electroweak scale in a natural way. Many aspects of the brane world scenario have been investigated: for example, the effective four-dimensional Einstein's equation on a positive tension brane [3], weak gravity [4,5], black holes [6], inflating branes [7], cosmologies [8–12], and so on.

In the original Randall-Sundrum brane world scenario, these authors considered five-dimensional pure gravity described by the Einstein-Hilbert action with a negative cosmological constant and two 3-branes with tension. Because of the negative cosmological constant the five-dimensional bulk geometry is highly curved and the curvature scale is possibly of the order of the five-dimensional Planck scale, while the induced geometries on the branes are flat, provided that brane tensions are fine-tuned. Therefore, it is expected that quantum effects in the bulk may be important in the brane world scenario, since quantum effects in curved spacetime usually become important when the spacetime geometry is highly curved or when the causal structure is non-trivial [13]. In this connection, several authors investigated quantum effects in the brane world scenario [14–18].

In particular, in ref. [16] exact semiclassical solutions representing a static brane world with two branes were obtained by analyzing the semiclassical Einstein's equation in five-dimensions with a negative cosmological constant and conformally invariant bulk matter fields. There, the following two types of solutions were found. Type-(a): solution with a positive tension brane and a negative tension brane. Type-(b): solution with two positive tension branes. For each type of semiclassical solution, two relations between the warp factor and brane tensions were found: one giving the warp factor as a function of the brane tensions and another giving a relation between the brane tensions.

Although it is interesting that we could obtain analytic solutions in the model of ref. [16], it seems that this model is not realistic enough. As far as the author knows, there is no realization of conformally invariant bulk matter fields starting from M-theory or superstring theory. Nonetheless, it is expected that the solutions in ref. [16] may actually capture some important features of quantum effects in the brane world scenario. Hence, it is worth while to extend the analysis of ref. [16] to more realistic brane world models which also take bulk quantum effects into account. For this purpose, one would like to consider higher curvature corrections to the tree-level bulk action. As discussed in the next section, R^4 corrections are realistic from the point view of M-theory.

In this paper a simple five-dimensional brane world model is proposed, motivated by M-theory compactified on a six-dimensional manifold of small radius and an S^1/Z_2 of large radius. We include the leading-order higher curvature correction to the tree-level bulk action. As a tractable model of the bulk theory we consider pure gravity including a (Ricci-scalar)^4-correction to the Einstein-Hilbert action. In this model theory, after a conformal transformation to the Einstein frame, we numerically obtain static solutions, each of which consists of a positive tension brane and

a negative tension brane. For these solutions, we obtain two relations between the warp factor and brane tensions. Those solutions and relations are a close analogue of the type-(a) solutions and relations obtained in the model of ref. [16]. On the other hand, in the present model it will be shown that there is no analogue of the type-(b) solutions. This fact might be considered to be consistent with the suggestion of refs. [14,18] that the type-(b) solutions are unstable.

This paper is organized as follows. In Sec. II we describe a simple brane world model which take bulk quantum effects into account. In Sec. III we numerically obtain static solutions in the model, and two relations between the warp factor and brane tensions are derived. Sec. IV is devoted to a summary of this paper.

II. MODEL DESCRIPTION

In this section we propose a simple brane world model, motivated by M-theory compactified on a six-dimensional compact manifold (eg. Calabi-Yau manifold) of small radius and an S^1/Z_2 of large radius [19]. In this situation, effectively we may consider a five-dimensional theory compactified on the S^1/Z_2. In the five-dimensional bulk action, we shall consider a correction by a R^4 term to the Einstein-Hilbert term [1] since several calculations of higher-order corrections to the effective action suggest that in eleven-dimensions R^2 terms do not appear but R^4 terms may appear [21,22] [2]. It is expected that the R^4 corrections play important roles in brane world scenarios since the curvature scale in the bulk may be comparable to the five-dimensional Planck scale and, thus, higher curvature corrections cannot be neglected.

On the other hand, as for the action on branes, higher curvature corrections are expected to be less important than those in the bulk and can be neglected since curvature scale induced on branes (at least on our brane) should be small compared to the Planck scale in low energy. Nonetheless, motivated by the four- and ten-dimensional effective theory induced on the fixed points of S^1/Z_2 [19,24], we may include R^2 corrections to brane actions. In the following, we will explicitly see that the R^2 corrections do not play any roles in our analysis of static solutions.

Since general higher curvature terms are difficult to analyze, for simplicity, we shall consider Ricci scalars only. Furthermore, we assume that the compactification from eleven dimensions to five dimensions is properly stabilized and, for simplicity again, we do not consider the corresponding moduli as dynamical fields in five dimensions. Namely, in our analysis, we shall consider the following action.

$$I = \int_M d^5x \sqrt{-g} \left[\frac{1}{2\kappa^2} R_5 + a\kappa^2 R_5^4 - \Lambda \right] + \int_\Sigma d^4y \sqrt{-q}(bR_4^2 - \lambda) + \int_{\bar{\Sigma}} d^4\bar{y} \sqrt{-\bar{q}}(\bar{b}\bar{R}_4^2 - \bar{\lambda}), \qquad (1)$$

where κ and Λ are the five-dimensional gravitational constant and cosmological constant, a, b and $\bar{b}$ are dimensionless constants, and λ and $\bar{\lambda}$ are brane tensions. The fixed-point hypersurface, or the world volume of a 3-brane, Σ (or $\bar{\Sigma}$) is represented by $x^M = Z^M(y)$ (or $x^M = \bar{Z}^M(\bar{y})$, respectively) and the induced metric $q_{\mu\nu}$ (or $\bar{q}_{\mu\nu}$, respectively) is defined by

$$q_{\mu\nu}(y) = e_\mu^M(y)e_\nu^N(y)g_{MN}|_{x=Z(y)},$$
$$e_\mu^M(y) = \frac{\partial Z^M(y)}{\partial y^\mu} \qquad (2)$$

(or $\bar{q}_{\mu\nu}(\bar{y}) = \bar{e}_\mu^M(\bar{y})\bar{e}_\nu^N(\bar{y})g_{MN}|_{x=\bar{Z}(\bar{y})}$, $\bar{e}_\mu^M(\bar{y}) = \partial\bar{Z}^M(\bar{y})/\partial\bar{y}^\mu$, respectively). The Ricci scalars R_5, R_4 and $\bar{R}_4$ are of g_{MN}, $q_{\mu\nu}$ and $\bar{q}_{\mu\nu}$, respectively.

Following ref. [25], we perform the conformal transformation

$$\hat{g}_{MN} = e^{(1/\sqrt{3})\kappa\psi}g_{MN},$$
$$\kappa\psi = \frac{2}{\sqrt{3}}\ln(1 + 8a\kappa^4 R_5^3) \qquad (3)$$

to obtain the following expression.

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[1] We, off course, include the Einstein-Hilbert term since it appears in the tree-level effective action in eleven dimensions [20].

[2] The importance of R^4 terms in M-theory was originally pointed out in ref. [21] and many authors showed evidences of it [22]. Possible cosmological consequences were discussed in ref. [23].

$$I = \int_M d^5x \sqrt{-\hat{g}} \left[\frac{\hat{R}_5}{2\kappa^2} - \frac{1}{2}\hat{g}^{MN}\partial_M\psi\partial_N\psi - U(\psi) \right]$$

$$+ \int_\Sigma d^4y \sqrt{-\hat{q}} \left[b\left(\hat{R}_4 + \sqrt{3}\kappa\hat{D}^2\psi - \frac{1}{2}\kappa^2\hat{q}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \right)^2 - f(\psi) \right]$$

$$+ \int_{\bar{\Sigma}} d^4\bar{y} \sqrt{-\hat{\bar{q}}} \left[\bar{b}\left(\hat{\bar{R}}_4 + \sqrt{3}\kappa\hat{\bar{D}}^2\psi - \frac{1}{2}\kappa^2\hat{\bar{q}}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \right)^2 - \bar{f}(\psi) \right], \tag{4}$$

where the conformally transformed induced metrics $\hat{q}_{\mu\nu}$ and $\hat{\bar{q}}_{\mu\nu}$ are defined by

$$\hat{q}_{\mu\nu}(y) = e^{(1/\sqrt{3})\kappa\psi(Z(y))}q_{\mu\nu}(y) = e_\mu^M(y)e_\nu^N(y)\hat{g}_{MN}|_{x=Z(y)},$$

$$\hat{\bar{q}}_{\mu\nu}(\bar{y}) = e^{(1/\sqrt{3})\kappa\psi(\bar{Z}(\bar{y}))}\bar{q}_{\mu\nu}(\bar{y}) = \bar{e}_\mu^M(\bar{y})\bar{e}_\nu^N(\bar{y})\hat{g}_{MN}|_{x=\bar{Z}(\bar{y})}, \tag{5}$$

$\hat{D}$ and $\hat{\bar{D}}$ are four-dimensional covariant derivatives compatible with $\hat{q}_{\mu\nu}$ and $\hat{\bar{q}}_{\mu\nu}$, respectively, and the Ricci scalars $\hat{R}_5$, $\hat{R}_4$ and $\hat{\bar{R}}_4$ are of $\hat{g}_{MN}$, $\hat{q}_{\mu\nu}$ and $\hat{\bar{q}}_{\mu\nu}$, respectively. The potential $U(\psi)$ and the functions $f(\psi)$ and $\bar{f}(\psi)$ are given by

$$U(\psi) = e^{-(5\sqrt{3}/6)\kappa\psi} \left[\Lambda + (3/16)\kappa^{-10/3}a^{-1/3}(e^{(\sqrt{3}/2)\kappa\psi} - 1)^{4/3} \right],$$

$$f(\psi) = e^{-(2/\sqrt{3})\kappa\psi}\lambda,$$

$$\bar{f}(\psi) = e^{-(2/\sqrt{3})\kappa\psi}\bar{\lambda}. \tag{6}$$

To obtain the expression (4) we have assumed that $8\kappa^4 aR_5^3 > -1$. For negative Λ, the potential can be rewritten as

$$U(\psi) = |\Lambda|e^{-(5\sqrt{3}/6)\kappa\psi} \left[-1 + \alpha(e^{(\sqrt{3}/2)\kappa\psi} - 1)^{4/3} \right], \tag{7}$$

where

$$\alpha = (3/16)\kappa^{-10/3}a^{-1/3}|\Lambda|^{-1}. \tag{8}$$

Since the $\hat{g}$-dependent part of the action (4) is of the Einstein-Hilbert form, the conformal frame in which the metric is $\hat{g}_{MN}$ is called *Einstein frame*. On the other hand, we shall call another conformal frame in which the metric is g_{MN} *the original frame*.

In this paper, we assume that $\Lambda < 0$ and consider a static configuration with the ansatz

$$\hat{g}_{MN}dx^M dx^N = e^{-2A(w)}\eta_{\mu\nu}dx^\mu dx^\nu + dw^2,$$

$$\psi = \psi(w). \tag{9}$$

This ansatz represents a general configuration with the four-dimensional Poincaré invariance. Off course, the set of all configurations with the four-dimensional Poincaré invariance in the Einstein frame is equivalent to that in the original frame. With this ansatz the curvature-squared term in the brane action does not contribute to the equation of motion at all. Einstein's equation and the field equation of the field ψ are given by

$$3\frac{d^2A}{dw^2} - \kappa^2\left(\frac{d\psi}{dw}\right)^2 = 0,$$

$$6\left(\frac{dA}{dw}\right)^2 - \kappa^2\left[\frac{1}{2}\left(\frac{d\psi}{dw}\right)^2 - U(\psi)\right] = 0,$$

$$e^{4A}\frac{d}{dw}\left(e^{-4A}\frac{d\psi}{dw}\right) - U'(\psi) = 0. \tag{10}$$

Note that the third equation is dependent of the first two equations unless $d\psi/dw = 0$ (the Bianchi identity), while the first equation can also result from the second and the last equations unless $dA/dw = 0$. When we compactify the w-direction by S^1/Z_2 so that $w \sim w + 2L$ and that $w \sim -w$, there appears the following matching condition for ψ.

$$2 \lim_{w \to +0} \frac{d\psi}{dw} = f'(\psi)\big|_{w=0},$$

$$2 \lim_{w \to L-0} \frac{d\psi}{dw} = - \bar{f}'(\psi)\big|_{w=L}, \tag{11}$$

where we have assumed that Σ and $\bar{\Sigma}$ are world-volumes of branes at the two fixed points $w = 0$ and $w = L$, respectively. We suppose that the brane at $w = 0$ is our world and shall call it *our brane*. We shall call another brane at $w = L$ *the hidden brane*. As for the function $A(w)$ in the metric, we have the junction condition

$$6 \lim_{w \to +0} \frac{dA}{dw} = \kappa^2 f(\psi)\big|_{w=0},$$

$$6 \lim_{w \to L-0} \frac{dA}{dw} = - \kappa^2 \bar{f}(\psi)\big|_{w=L}. \tag{12}$$

This is a special case of Israel's junction condition [26]. In the Einstein frame the so called warp factor can be defined by $\phi_E = e^{A(0)}/e^{A(L)}$. Correspondingly, the warp factor in the original frame is

$$\phi = \exp\left\{ [A(0) - A(L)] + \frac{\kappa}{2\sqrt{3}}[\psi(0) - \psi(L)] \right\}. \tag{13}$$

It is evident that $\psi \equiv \psi_0$ is not a solution because of the matching condition (11), where ψ_0 is an extremum of the potential $U(\psi)$. In particular, we can show that

$$\lim_{w \to +0} U(\psi) = \lim_{w \to L-0} U(\psi) = 0, \tag{14}$$

and thus ψ cannot stay at ψ_0 unless Λ is zero. Actually, provided that equations of motion (10) and the junction condition (12) are satisfied, the matching condition (11) is equivalent to the vanishing-potential condition (14) combined with

$$\lim_{w \to +0} \frac{dA}{dw} \cdot \frac{d\psi}{dw} \le 0,$$

$$\lim_{w \to L-0} \frac{dA}{dw} \cdot \frac{d\psi}{dw} \le 0. \tag{15}$$

In order to see the necessity of the condition (15), notice that $f(\psi)f'(\psi) \le 0$ and $\bar{f}(\psi)\bar{f}'(\psi) \le 0$.

III. NUMERICAL SOLUTION

For the purpose of numerical integration, it is convenient to rewrite all equations in terms of dimensionless variables. Hence, we introduce the dimensionless independent variable x defined by $x = w/L$ and consider the region $0 \le x \le 1$, where L is the distance between two branes. As for the dependent variables, we introduce the following three:

$$y_1(x) \equiv A,$$

$$y_2(x) \equiv L \frac{dA}{dw},$$

$$y_3(x) \equiv \frac{\kappa}{2\sqrt{3}}\psi. \tag{16}$$

Differential equations for these dimensionless independent variables are given by

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = 4[y_2^2 + (L/l)^2 V(y_3)],$$

$$(\dot{y}_3)^2 = y_2^2 + (L/l)^2 V(y_3), \tag{17}$$

where dots denote differentiation with respect to x, the length scale l is defined by $l = \kappa^{-1}\sqrt{6/|\Lambda|}$, and

$$V(y_3) = e^{-5y_3}\left[-1 + \alpha(e^{3y_3} - 1)^{4/3} \right]. \tag{18}$$

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As already mentioned in the previous section we assume that $\Lambda < 0$. The set of these three differential equations is equivalent to the equation of motion (10) as long as $\dot{y}_2$ is not zero. As we shall argue later, there is no static solution for $\alpha \leq 1$. Hence, for the present we shall concentrate on the case $\alpha > 1$ only. The potential is shown in Figure 1 for $\alpha = 2.0$, 1.5, and 1.2. The vanishing-potential condition (14) is written as

$$V(y_3(0)) = V(y_3(1)) = 0, \tag{19}$$

and should be complemented by

$$\begin{aligned} y_2(0)\dot{y}_3(0) &\leq 0, \\ y_2(1)\dot{y}_3(1) &\leq 0. \end{aligned} \tag{20}$$

It is easy to impose the boundary condition (19) since the roots of $V(y_3)$ are analytically obtained as $y_3 = y_\pm$ for $\alpha > 1$, where

$$y_\pm = \frac{1}{3} \ln(1 \pm \alpha^{-3/4}). \tag{21}$$

The complementary condition (20) should be checked after a solution of the differential equation (17) with the boundary condition (19) is obtained. Thence, the junction condition (12) determines the brane tensions as

$$\begin{aligned} \lambda/(6\kappa^{-2}l^{-1}) &= (l/L)y_2(0)e^{4y_3(0)}, \\ \bar{\lambda}/(6\kappa^{-2}l^{-1}) &= -(l/L)y_2(1)e^{4y_3(1)}. \end{aligned} \tag{22}$$

Finally, the warp factor ϕ given by (13) is written as

$$\phi = \exp\left[y_1(0) + y_3(0) - y_1(1) - y_3(1)\right]. \tag{23}$$
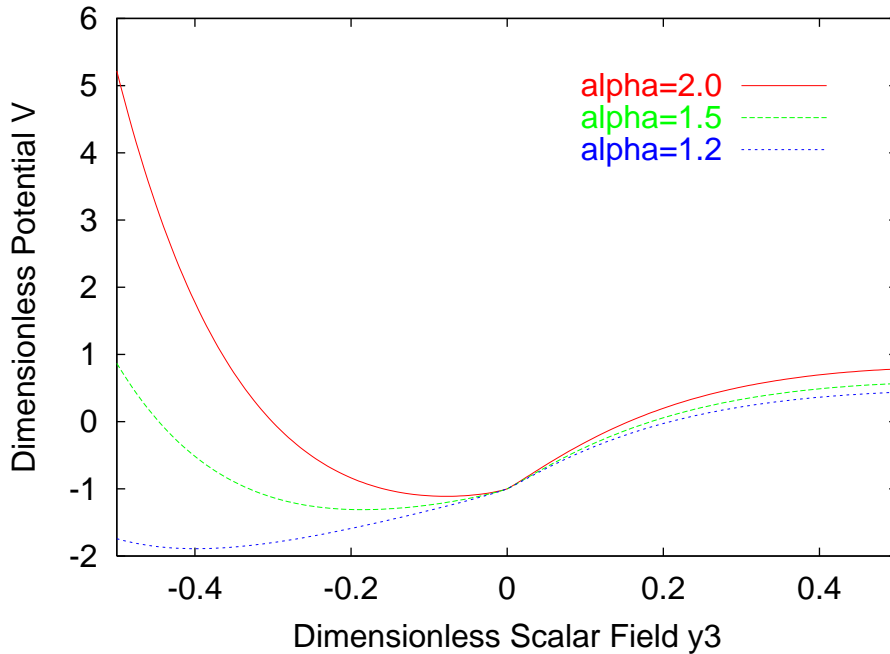

FIG. 1. For $\alpha > 1$ the dimensionless potential $V(y_3)$ has roots $y_3 = y_\pm$, where $y_\pm$ are given by (21), and a global minimum. In this figure, $V(y_3)$ is shown for $\alpha = 2.0$, 1.5, 1.2.

Note that, without loss of generality, we can impose the additional condition

$$y_1(0) = 0, \tag{24}$$

since none of the above equations is changed by the shift $y_1(x) \to y_1(x) - y_1(0)$. This additional condition combined with the vanishing-potential condition (19), which can be rewritten as

$$y_3(0) = y_\pm,$$
$$y_3(1) = y_\pm, \tag{25}$$

give enough number of boundary conditions for the set of three differential equations (17). Here, plus or minus signs in two of (25) are independent. According to four possible choices of the signs in (25), there are four possible types of solutions.

$$(++)\text{-type} : y_3(0) = y_+, \ y_3(1) = y_+,$$
$$(+-)\text{-type} : y_3(0) = y_+, \ y_3(1) = y_-,$$
$$(-+)\text{-type} : y_3(0) = y_-, \ y_3(1) = y_+,$$
$$(--)\text{-type} : y_3(0) = y_-, \ y_3(1) = y_-. \tag{26}$$

We shall solve the differential equation (17) with the boundary condition (24) and (25) by the so called relaxation method [27]. This method works very well if a good initial guess is given. In the following we shall solve the differential equation many times, each time with different values of the parameters α and L/l. Hence, the previous solution can be used as a good initial guess in the next calculation with slightly different parameters.

Figures 2, 3, 4, 5, 6 and 7 show several $(-+)$-type solutions obtained by the relaxation method. Physical parameters in these solutions are $\alpha = 2.0, 1.5, 1.2$ and $L/l = 10.0, 20.0, 30.0$.
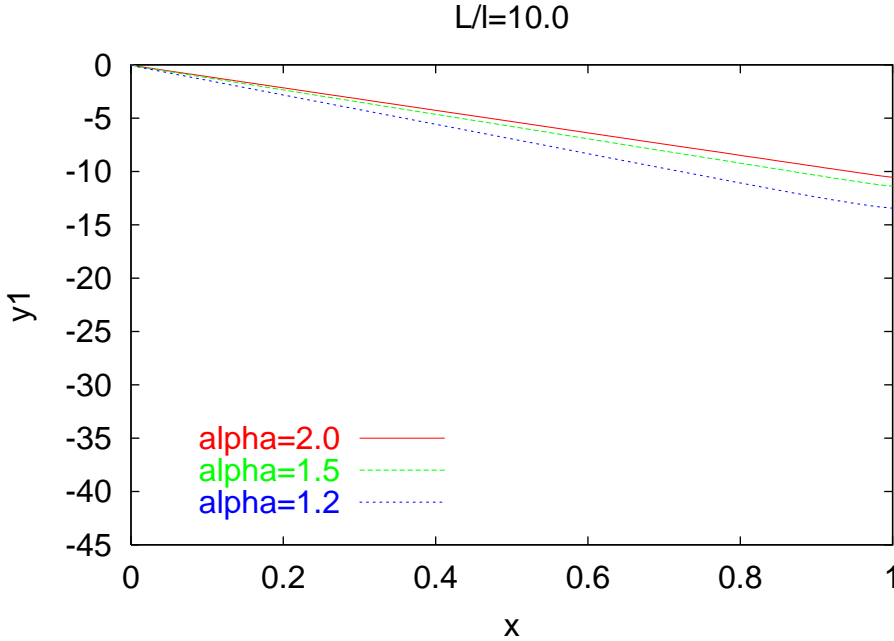

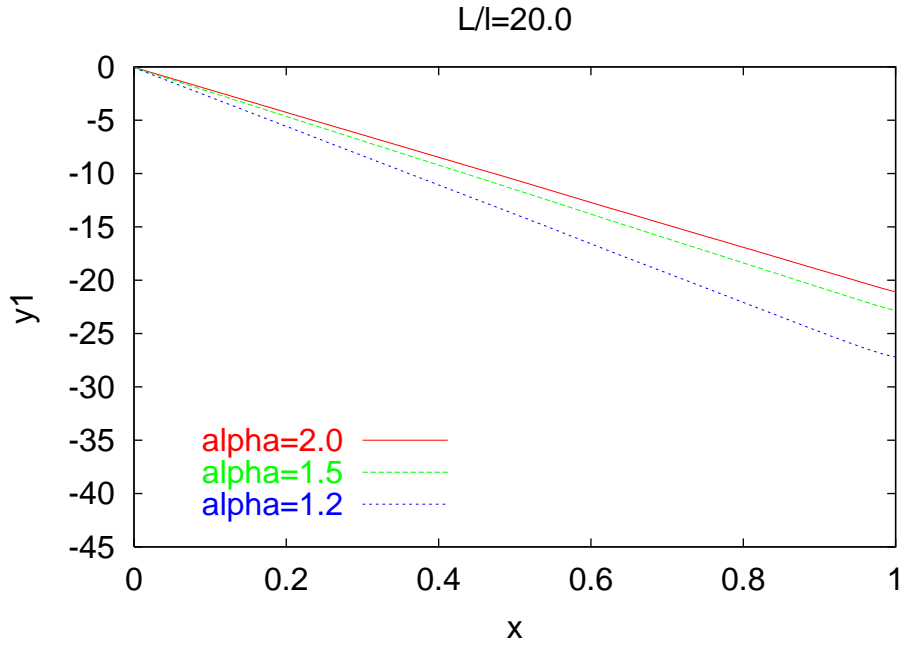FIG. 2. The numerical solution $y_1(x)$ for $L/l = 10.0$ and $\alpha = 2.0, 1.5, 1.2$.

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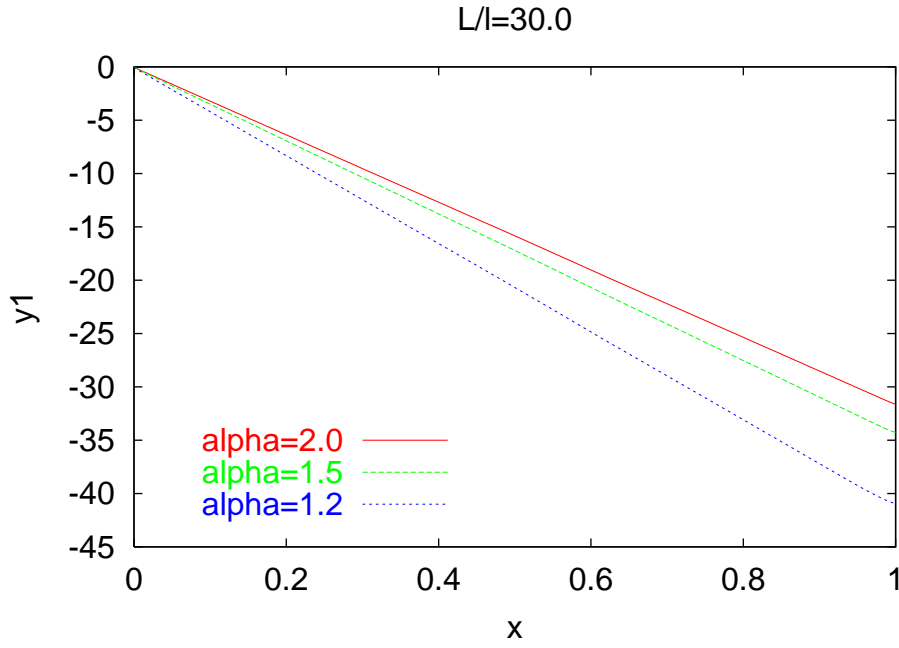FIG. 3. The numerical solution $y_1(x)$ for $L/l = 20.0$ and $\alpha = 2.0,\ 1.5,\ 1.2$.

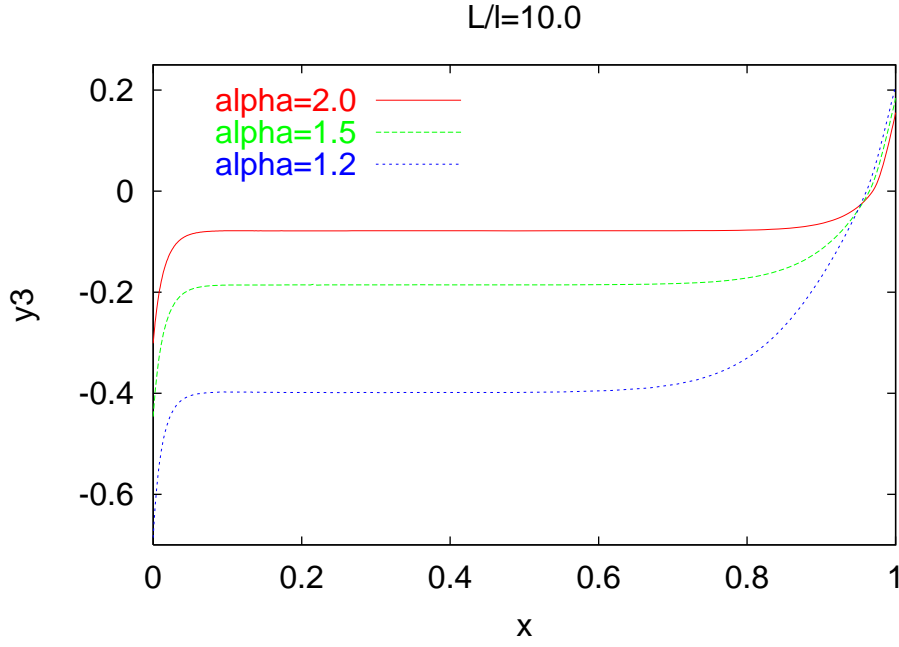FIG. 4. The numerical solution $y_1(x)$ for $L/l = 30.0$ and $\alpha = 2.0,\ 1.5,\ 1.2$.

FIG. 5. The numerical solution $y_3(x)$ for $L/l = 10.0$ and $\alpha = 2.0,\ 1.5,\ 1.2$.
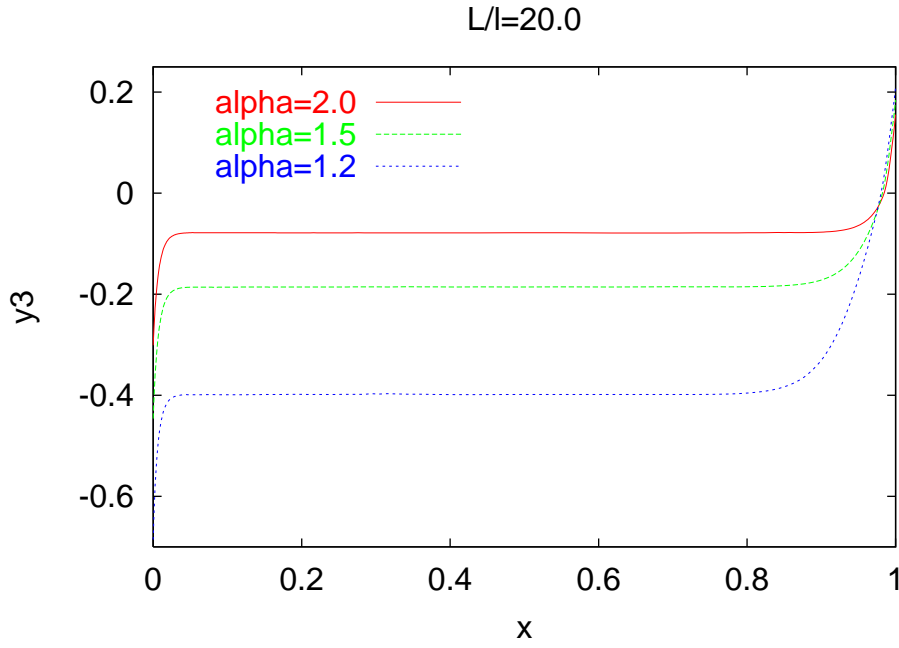

FIG. 6. The numerical solution $y_3(x)$ for $L/l = 20.0$ and $\alpha = 2.0,\ 1.5,\ 1.2$.
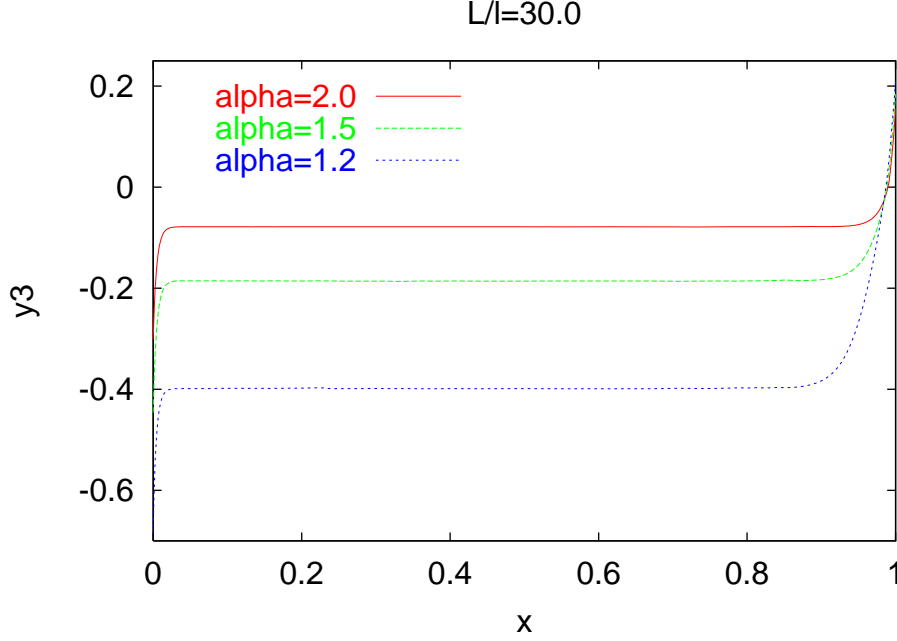
FIG. 7. The numerical solution $y_3(x)$ for $L/l = 30.0$ and $\alpha = 2.0$, 1.5, 1.2.

From these figures, we can see that, except for vicinities of the boundaries, y_3 stays near the minimum y_{min} of the potential $V(y_3)$ and y_1 is almost linear in x: [3]

$$y_1(x) \simeq -\frac{L}{l}\sqrt{|V(y_{min})|}\, x,$$
$$y_3(x) \simeq y_{mim}. \tag{27}$$

In other words, except for vicinities of the boundaries, the scalar field ψ stays near the minimum of the potential $U(\psi)$ and the five-dimensional geometry is almost the AdS whose curvature is determined by the minimal value of $U(\psi)$. However, because of the boundary condition (25), near-boundary behaviors of solutions are non-trivial. Actually, figures 8 and 9 show that the five dimensional geometry near the boundaries deviates rather strongly from AdS. The deviation is larger for a smaller value of α, or a larger value of the coefficient a of the R^4-term, as easily expected. Note that $y_2 = \dot{y}_1$ would be independent of x if the geometry was AdS. Each of figures 10 and 11 shows how the scalar field approaches one of roots of the potential.

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[3] As we shall argue in the paragraph after the next, these approximate expressions could be inferred without any numerical calculations for large values of L/l.
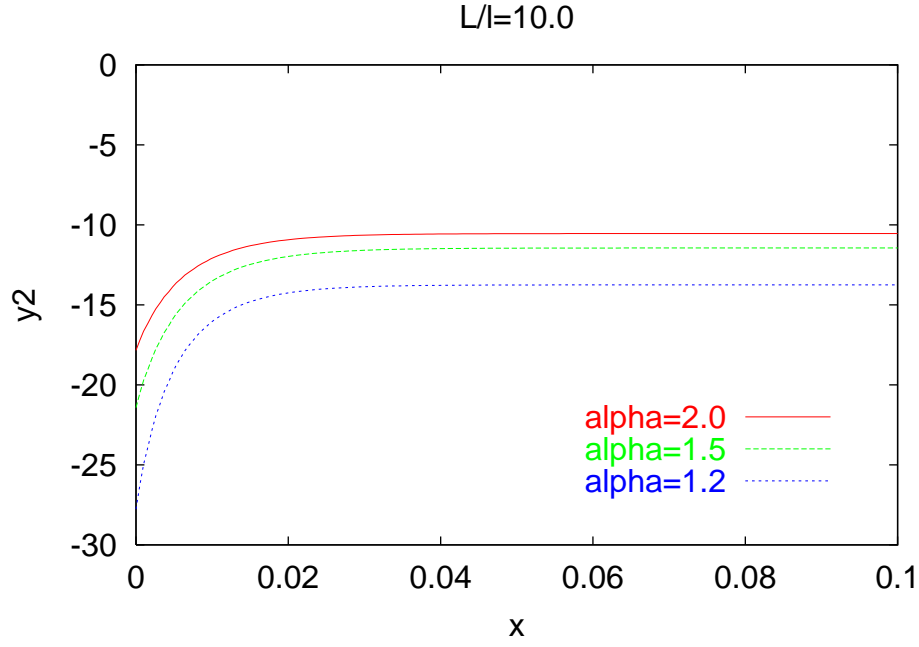
FIG. 8. The numerical solution $y_2(x) \equiv \dot{y}_1(x)$ in the vicinity of the boundary $x = 0$ for $L/l = 10.0$ and $\alpha = 2.0$, 1.5, 1.2.
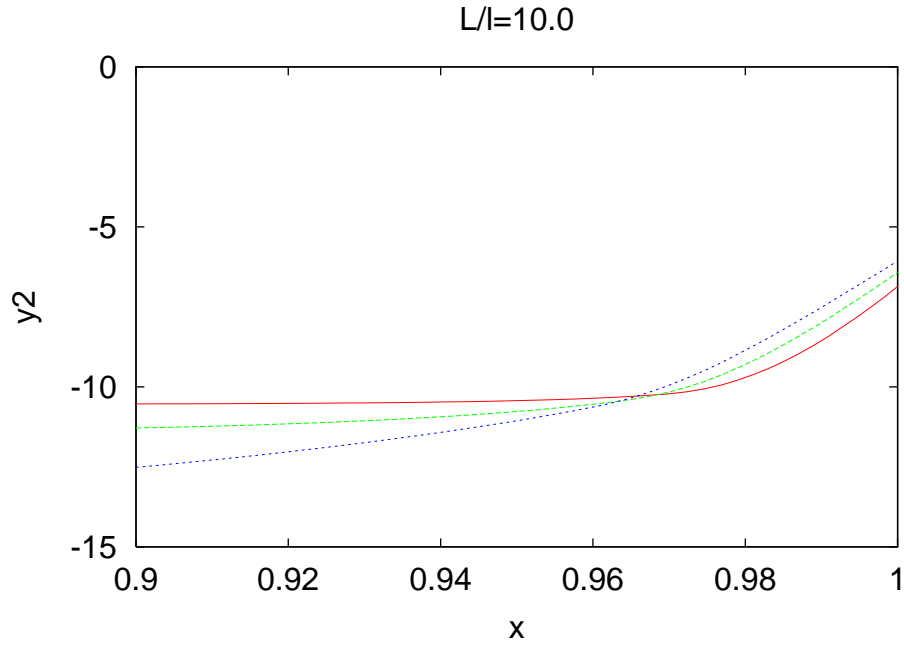

FIG. 9. The numerical solution $y_2(x) \equiv \dot{y}_1(x)$ in the vicinity of the boundary $x = 1$ for $L/l = 10.0$ and $\alpha = 2.0$, 1.5, 1.2.
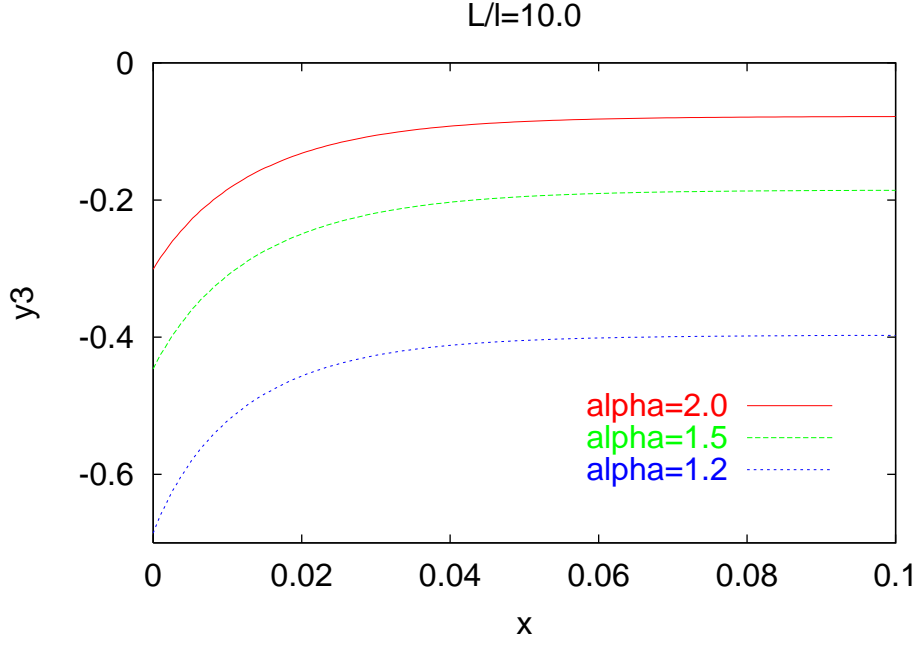
FIG. 10. The numerical solution $y_3(x)$ in the vicinity of the boundary $x = 0$ for $L/l = 10.0$ and $\alpha = 2.0$, 1.5, 1.2.
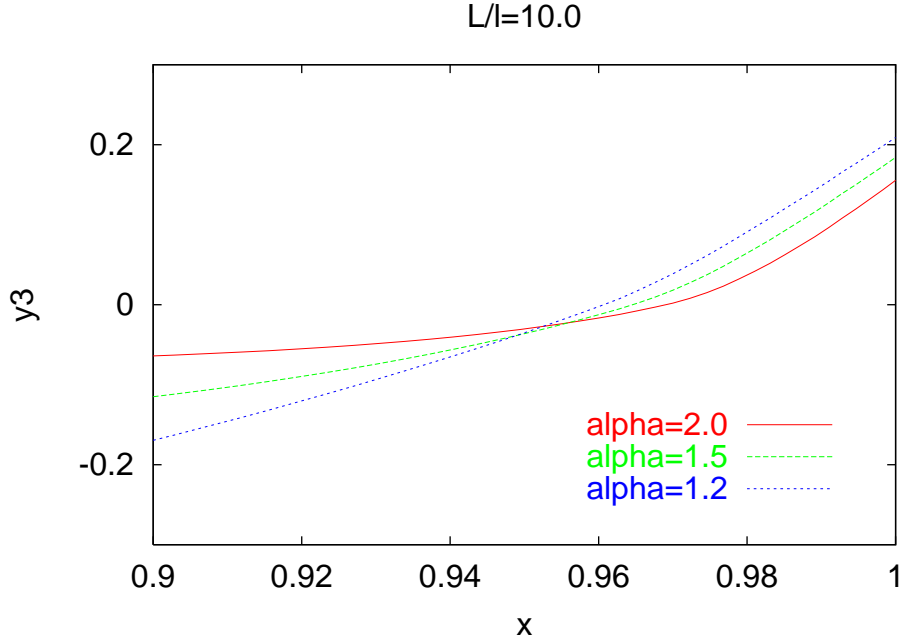

FIG. 11. The numerical solution $y_3(x)$ in the vicinity of the boundary $x = 1$ for $L/l = 10.0$ and $\alpha = 2.0$, 1.5, 1.2.

We now argue that there are no (static) solutions of $(++)$- and $(--)$-types. First, let us rewrite the differential equation as

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = 4y_4^2,$$

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$$\dot{y}_3 = y_4,$$
$$\dot{y}_4 = 4y_2y_4 + (L/l)^2 V'(y_3)/2, \tag{28}$$

where

$$y_4(x) = \frac{\kappa}{2\sqrt{3}} L \frac{d\psi}{dw}, \tag{29}$$

and the corresponding boundary condition is

$$y_1(0) = 0,$$
$$y_2(0) + y_4(0) = 0,$$
$$y_3(0) = y_\pm,$$
$$y_2(1) + y_4(1) = 0. \tag{30}$$

It is easy to show from (28) and (30) that

$$\int_0^1 dx\, e^{-4y_3} V'(y_3) = 0. \tag{31}$$

Next, it is also easy to show by using the last equation of (28) that, if $y_4(x_1) \leq 0$ and $y_3(x_1) < y_{min}$ for $0 < {}^\exists x_1 < 1$, then $y_4(x) \leq 0$ for $x_1 \leq {}^\forall x \leq 1$, where y_{min} is the global minimum of $V(y_3)$ between y_- and y_+. Thus, if y_3 starts from y_+ at $x = 0$ and reaches the region $y_3 < y_{min}$ then y_3 cannot return to y_+ at $x = 1$. Thirdly, combining this fact with (31), we can show that there is no solution of the $(++)$-type which is bounded in the region $y_- \leq y_3 \leq y_+$. Actually, (31) implies that if y_3 starts from y_+ at $x = 0$ and is bounded in the region $y_- \leq y_3 \leq y_+$ then y_3 should reach the region $y_3 < y_{min}$, from which y_3 cannot return to y_+ at $x = 1$. Similarly, we can show that there is no solution of the $(--)$-type which is bounded in the region $y_- \leq y_3 \leq y_+$. Heuristically, the above mathematical statement can be easily inferred if one interprets the last two equations of (28) as the equation of motion for a particle position $y_3(x)$ at time x. In this interpretation, the particle receives the force due to the reversed potential $-(L/l)^2 V(y_3)/2$ and the friction (or anti-friction) force $4y_2\dot{y}_3$. Moreover, with this heuristic interpretation in mind, it seems likely that any solution should be bounded in the region $y_- \leq y_3 \leq y_+$ since the reversed potential $-(L/l)^2 V(y_3)/2$ is negative outside the region $y_- \leq y_3 < y_+$ and $V(y_3(0)) = V(y_3(1)) = 0$. Finally, combining this with the above mathematical statement, we can say at least for moderate α that there are no solutions of $(++)$- and $(--)$-types.

Moreover, $(+-)$- and $(-+)$-types are equivalent to each other up to the coordinate transformation $x \to 1 - x$. Hence, $(-+)$-type solutions are all we have been seeking.

Figures 12 and 13 show relations between the warp factor ϕ given by (23) and the brane tension λ or $\bar{\lambda}$ given by (22) for $(-+)$-type solutions. We can see that the tension of each brane converges quickly to an α-dependent value when the warp factor becomes large. To be precise,

$$\lambda/(6\kappa^{-2}l^{-1}) \to \begin{cases} -0.54 & (\alpha = 2.0) \\ -0.36 & (\alpha = 1.5) \\ -0.18 & (\alpha = 1.2) \end{cases}, \tag{32}$$

$$\bar{\lambda}/(6\kappa^{-2}l^{-1}) \to \begin{cases} 1.28 & (\alpha = 2.0) \\ 1.34 & (\alpha = 1.5) \\ 1.40 & (\alpha = 1.2) \end{cases} \tag{33}$$

as $\phi \to \infty$. The convergence can be easily understood as follows by the heuristic interpretation. We consider a particle in one dimension whose position at time x is $y_3(x)$ and which receives a force due to the reversed potential $-(L/l)^2 V(y_3)/2$ and the friction (or anti-friction) force. The particle moves from $y_3 = y_-$ to $y_3 = y_+$ in the fixed duration $0 \leq x \leq 1$. For a large L/l, in order to satisfy this boundary condition, the particle should stay near $y_3 = y_{min}$ for a relatively long time since the reversed potential is steep. In this case, the initial (or final) velocity $y_4(0)$ (or $y_4(1)$, respectively) should be fine-tuned to a value close to the 'escape velocity', which is roughly proportional to L/l, against the reversed potential. Because of the relation $y_2(0) + y_4(0) = 0$ (or $y_2(1) + y_4(1) = 0$, respectively) and the junction condition (22), the fine-tuning of $y_4(0)$ (or $y_4(1)$, respectively) is equivalent to the fine-tuning of λ (or $\bar{\lambda}$, respectively). The required value of λ (or $\bar{\lambda}$, respectively) is almost independent of L/l since the required value of $y_4(0)$ (or $y_4(1)$, respectively) is roughly proportional to L/l as stated above. On the other hand, when y_3 stays near y_{min} with a very small velocity, y_2 should satisfy $y_2^2 \approx -(L/l)^2 V(y_{min})$ since the right hand side of the last equation of (17) should vanish approximately. Hence, y_1 grows approximately linearly in x with the growth rate

proportional to L/l when y_3 stays near y_{min}. Thus, the exponent of the warp factor should be roughly proportional to L/l. Actually, except for vicinities of the boundaries, the numerical solutions are well approximated by (27) and thus the warp factor is well approximated by

$$\phi \simeq \exp\left[\frac{L}{l}\sqrt{|V(y_{min})|} + y_- - y_+\right]. \tag{34}$$

Finally, combining the approximate linearity of the exponent of the warp factor with the fine-tuning of the brane tension, we can conclude that the brane tension should converge to a constant as the warp factor becomes large. The limiting value of the brane tension should depend on the parameter α since the 'escape velocity' depends on α. This conclusion is of course consistent with the numerical result.

Figure 14 shows a relation between tension of our brane at $x = 0$ and that of the hidden brane at $x = 1$. This relation can be considered as a necessary condition for the system with two branes to be static, or the condition for the four-dimensional cosmological constant to vanish.
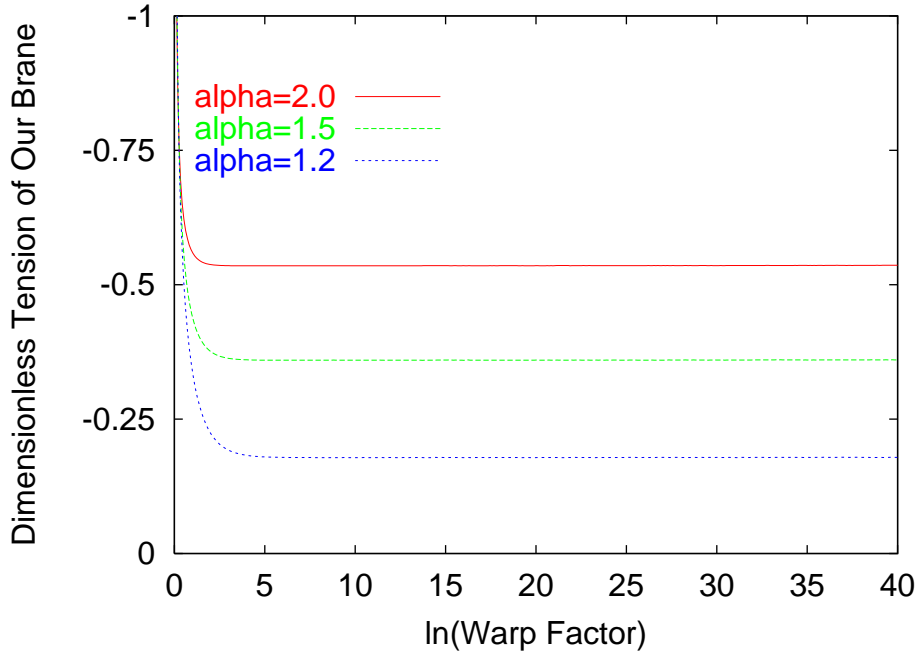

FIG. 12. The relation between the warp factor ϕ given by (23) and the tension λ of our brane at $x = 0$, which is given by (22), for the $(-+)$-type solutions. The horizontal axis represents $\ln\phi$ and the vertical axis represents $\lambda/(6\kappa^{-2}l^{-1})$. The physical parameter is $\alpha = 2.0, 1.5, 1.2$. As ϕ becomes large, $\lambda/(6\kappa^{-2}l^{-1})$ converges quickly to the α-dependent value given by (32).
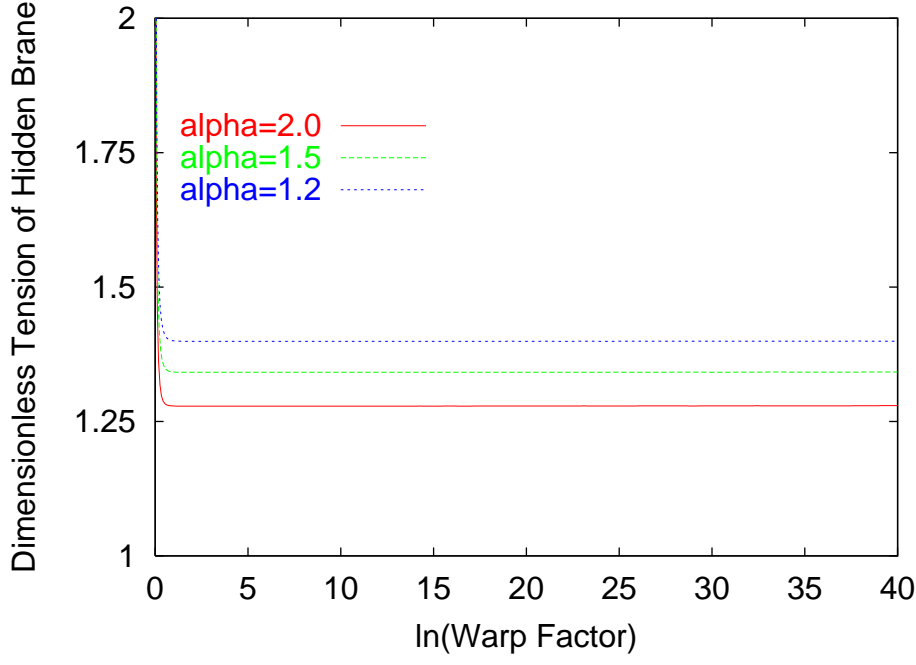
FIG. 13. The relation between the warp factor ϕ given by (23) and the tension $\bar{\lambda}$ of the hidden brane at $x = L$, which is given by (22), for the $(-+)$-type solutions. The horizontal axis represents $\ln \phi$ and the vertical axis represents $\bar{\lambda}/(6\kappa^{-2}l^{-1})$. The physical parameter is $\alpha = 2.0, 1.5, 1.2$. As ϕ becomes large, $\bar{\lambda}/(6\kappa^{-2}l^{-1})$ converges quickly to the α-dependent value given by (33).
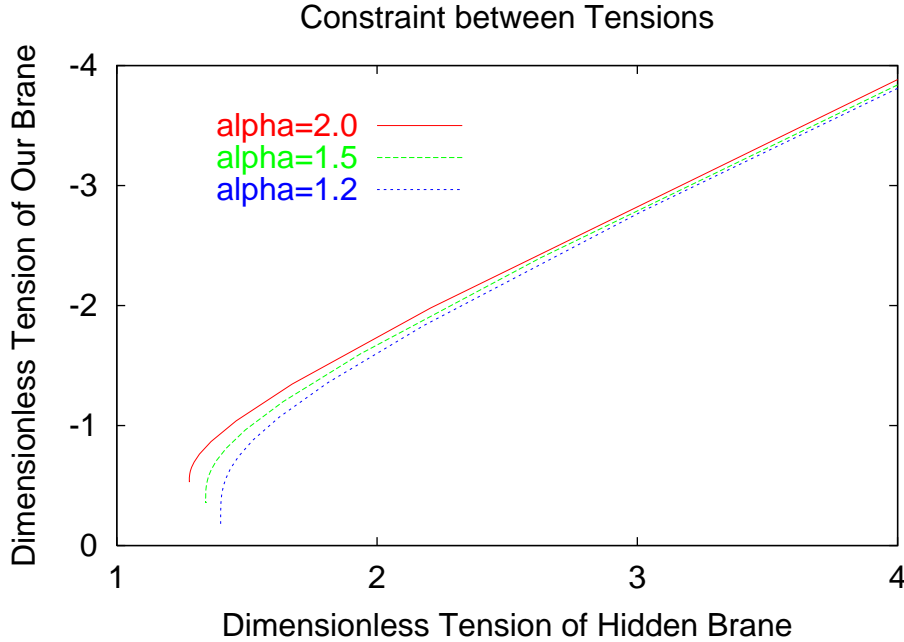

FIG. 14. The relation between tension λ of our brane at $x = 0$ and the tension $\bar{\lambda}$ of the hidden brane at $x = 1$. The horizontal axis represents $\bar{\lambda}/(6\kappa^{-2}l^{-1})$ and the vertical axis represents $\lambda/(6\kappa^{-2}l^{-1})$. This relation can be considered as a necessary condition for the system with two branes to be static, or the condition for the four-dimensional cosmological constant to vanish. The physical parameter is $\alpha = 2.0, 1.5, 1.2$.

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Now let us give an argument that there is no static solution for $\alpha \leq 1$. The following argument is almost the same as that for the non-existence of $(++)$- and $(--)$-type solutions. First, let us consider the case $0 < \alpha \leq 1$. In this case the potential $V(y_3)$ has only one root $y = y_+$, and thus only $(++)$-type solutions are allowed if any. However, since $V'(y_3) > 0$ for $y_3 \leq y_+$, the condition (31) excludes $(++)$-type solutions which are bounded in the region $y_3 \leq y_+$. On the other hand, since the reversed potential $-(L/l)^2 V(y_3)/2$ is negative for $y_3 > y_+$ and $V(y_3(0)) = V(y_3(1)) = 0$, the above heuristic interpretation in terms of a particle motion strongly suggests that, if there exists a solution of the differential equation, the solution should be bounded in the region $y_3 \leq y_+$. Hence, we can conclude that there is no solution for $0 < \alpha \leq 1$. Next, for $\alpha \leq 0$, there is no root of the potential $V(y_3)$. Hence, in this case there is no way to satisfy the boundary condition (25). Therefore, there is no solution for $\alpha \leq 1$, and it is actually enough to concentrate on the case $\alpha > 1$ as we did.

IV. SUMMARY AND DISCUSSIONS

We have proposed a simple five-dimensional brane world model, motivated by M-theory compactified on a six-dimensional manifold of small radius and an S^1/Z_2 of large radius. We have included the leading-order higher curvature correction to the tree-level bulk action since in brane world scenarios the curvature scale in the bulk may be comparable to the five-dimensional Planck scale and, thus, higher curvature corrections may become important. As a manageable model of the bulk theory we have considered pure gravity including a (Ricci-scalar)^4-correction to the Einstein-Hilbert action.

In this model theory, after a conformal transformation to the Einstein frame, we have numerically obtained static solutions, each of which consists of a positive tension brane and a negative tension brane. The solutions are parameterized by a dimensionless parameter α in the bulk theory and L/l, where L is distance between two branes and l is a length scale determined by the (negative) bulk cosmological constant. Several solutions are shown in Figures 2, 3, 4, 5, 6 and 7.

The warp factor and tension of both branes have been calculated for various values of α and L/l and, by eliminating L/l, we have obtained two α-dependent relations between the warp factor and brane tensions. The existence of these relations implies that, contrary to the original Randall-Sundrum model, the so called radion is no longer a zero mode. In this sense, the present model is similar to those in Refs. [28,29,5]. The two relations completely determine the brane tensions as functions of the warp factor and are shown in Figures 12 and 13. From these figures we conclude that the tension of our brane should be negative and that fine-tuning of the tension of both branes is necessary for a large warp factor to explain the large hierarchy between the Planck scale and the electroweak scale. To be precise, the brane tensions should be fine-tuned with high accuracy to values shown in equations (32) and (33).

Further, eliminating the warp factor from Figures 12 and 13, we have obtained a relation between the brane tensions. It is shown in Figure 14 and can be considered as a necessary condition for the system with two branes to be static, or the condition for the four-dimensional cosmological constant to vanish. Namely, unless this relation is satisfied, the system cannot be static but becomes dynamical, regardless of initial conditions (i.e. initial position of branes, initial velocity of branes, and so on).

A stability analysis of solutions obtained in the present paper is an important topic for future work. Here, we only offer a comment concerning this subject: we cannot derive a correct effective action by simply substituting the solutions into the action. Actually, if we substitute any static solutions into the action then the action vanishes because of the Hamiltonian constraint. A simple illustration of this fact is given in Appendix A.

Several extensions of the present model may also be of interest for future work: (i) inflating brane solution; (ii) cosmological solution; (iii) inclusion of Ricci tensor and Weyl tensor contributions to the R^4-term in the action; (iv) inclusion of a 3-form field and modulus corresponding to the six-dimensional compactification. (i) It is probably not difficult to extend the static solutions in the present paper to inflating brane solutions. The relation between brane tensions, corresponding to Figure 14, is expected to become dependent on the four-dimensional cosmological constant induced on branes as well as the model parameter α. (ii) Extension to cosmological solutions should be possible. This should not be as difficult as extension of the semiclassical solutions in ref. [16] to cosmological solutions. The latter seems rather difficult because of the so called moving mirror effect [13], which is non-local. On the other hand, the present model has a locally defined Lagrangian density in five dimensions. Hence, extension to the cosmological context is easier in the present model than in the model of ref. [16]. Moreover, the present model seems more realistic and, thus, worth while investigating in more detail. (iii) Although we have investigated effects of the (Ricci-scalar)^4-correction only, it would be desirable to investigate effects of other forth-order curvature terms. Note, however, that effects of forth-order Weyl terms are probably less important insofar as we consider the metric (9) or small perturbations around it, since the Weyl tensor vanishes for the metric (9). (iv) In the present model we have considered pure gravity in the bulk. However, more realistic model should include a 3-form field existing in the bosonic sector of eleven dimensional

supergravity as well as moduli fields due to the compactification from eleven dimensions to five dimensions. In this case, we may consider effects due to various eighth-order derivative terms other than R^4 terms.

APPENDIX A: ESTIMATE OF THE ACTION

By the ansatz (9), the action (4) is reduced to

$$I = \int d^4x \mathcal{L},$$
$$\mathcal{L} = \oint dw e^{-4A} \left\{ \frac{2}{\kappa^2} \left[2\frac{d^2A}{dw^2} - 5\left(\frac{dA}{dw}\right)^2 \right] - \left[\frac{1}{2}\left(\frac{d\psi}{dw}\right)^2 + \tilde{U}(\psi) \right] \right\}, \tag{A1}$$

where the integration with respect to w in this expression is over the whole S^1, and

$$\tilde{U}(\psi) = U(\psi) + f(\psi)\delta(w) + \bar{f}(\psi)\delta(w - L). \tag{A2}$$

From this reduced action, the following equations of motion are derived.

$$3\left[2\left(\frac{dA}{dw}\right)^2 - \frac{d^2A}{dw^2} \right] + \kappa^2 \left[\frac{1}{2}\left(\frac{d\psi}{dw}\right)^2 + \tilde{U}(\psi) \right] = 0,$$
$$e^{4A}\frac{d}{dw}\left(e^{-4A}\frac{d\psi}{dw} \right) - \tilde{U}'(\psi) = 0. \tag{A3}$$

These are equivalent to equations (10), (11) and (12), provided that the identification $w \sim w + 2L \sim -L$ is imposed. We can estimate the value of the reduced action by using these equations. Actually, the first of (A3) reduces the effective Lagrangian density $\mathcal{L}$ to

$$\mathcal{L} = \frac{1}{\kappa^2} \oint dw e^{-4A} \left\{ 2\left[2\frac{d^2A}{dw^2} - 5\left(\frac{dA}{dw}\right)^2 \right] + 3\left[2\left(\frac{dA}{dw}\right)^2 - \frac{d^2A}{dw^2} \right] \right\}$$
$$= \frac{1}{\kappa^2} \oint dw \frac{d}{dw}\left(e^{-4A}\frac{dA}{dw} \right) = 0. \tag{A4}$$

Therefore, the effective Lagrangian density vanishes if a solution of equations of motion is substituted. This fact can be easily understood as follows [4]: in the static case without boundaries the Lagrangian of the system is just minus the Hamiltonian, which should vanish because of the Hamiltonian constraint.

The above arguments can be applied to the situation in ref. [28] by simply replacing $U(\psi)$, $f(\psi)$ and $\bar{f}(\psi)$ with appropriate functions. Since the above analysis indicates a vanishing effective potential, it is impossible to obtain a correct effective potential for the so called radion, which corresponds to L in the above arguments, by simply substituting solutions to the action. It seems that the non-vanishing effective potential obtained in ref. [28] merely measures an amount of inconsistency in their analysis. Nonetheless, their conjecture that the radion can be stabilized by inclusion of a bulk scalar field seems correct if the backreaction of the scalar field to the geometry is sufficiently small [5].

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